

THE ℓ^1 -NORM OF THE FOURIER TRANSFORM ON COMPACT VECTOR SPACES

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ABSTRACT. Suppose that G is a compact Abelian group. If $A \subset G$ then how small can $\|\chi_A\|_{A(G)}$ be? In general there is no non-trivial lower bound.

In [GK09] Green and Konyagin show that if \widehat{G} has sparse small subgroup structure and A has density α with $\alpha(1 - \alpha) \gg 1$ then $\|\chi_A\|_{A(G)}$ does admit a non-trivial lower bound.

In this paper we address the complementary case of groups with duals having rich small subgroup structure, specifically the case when G is a compact vector space over \mathbb{F}_2 . The results themselves are rather technical to state but the following consequence captures their essence: If $A \subset \mathbb{F}_2^n$ is a set of density as close to $\frac{1}{3}$ as possible then we show that $\|\chi_A\|_{A(\mathbb{F}_2^n)} \gg \log n$.

We include a number of examples and conjectures which suggest that what we have shown is very far from a complete picture.

1. NOTATION AND INTRODUCTION

We use the Fourier transform on compact Abelian groups, the basics of which may be found in Chapter 1 of Rudin [Rud90]; we take a moment to standardize our notation.

Suppose that G is a compact Abelian group. Write \widehat{G} for the dual group, that is the discrete Abelian group of continuous homomorphisms $\gamma : G \rightarrow S^1$, where $S^1 := \{z \in \mathbb{C} : |z| = 1\}$. Although the natural group operation on \widehat{G} corresponds to pointwise multiplication of characters we shall denote it by '+' in alignment with contemporary work. G may be endowed with Haar measure μ_G normalised so that $\mu_G(G) = 1$ and as a consequence we may define the Fourier transform $\widehat{\cdot} : L^1(G) \rightarrow \ell^\infty(\widehat{G})$ which takes $f \in L^1(G)$ to

$$\widehat{f} : \widehat{G} \rightarrow \mathbb{C}; \gamma \mapsto \int_{x \in G} f(x) \overline{\gamma(x)} d\mu_G(x).$$

We write

$$A(G) := \{f \in L^1(G) : \|\widehat{f}\|_1 < \infty\},$$

and define a norm on $A(G)$ by $\|f\|_{A(G)} := \|\widehat{f}\|_1$.

The following proposition is easy to prove and has been known at least since the 1960s.

Proposition 1.1. *Suppose that G is a compact vector space over \mathbb{F}_2 and $A \subset G$ has density (i.e. measure with respect to the probability measure μ_G) $1/3$. Then $\chi_A \notin A(G)$.*

This proposition only has content for infinite G ; in this paper we prove the following quantitative version which has content for finite G too.

Theorem 1.2. *Suppose that G is a compact vector space over \mathbb{F}_2 and $A \subset G$ has density α with $|\alpha - 1/3| \leq \epsilon$. Then*

$$\|\chi_A\|_{A(G)} \gg \log \log \epsilon^{-1}.$$

In view of Proposition 1.1 one could restrict attention in Theorem 1.2 to the case when G is finite rendering the compactness requirement irrelevant.

There is a simple construction, the details of which are given in §3, of a set A satisfying the hypotheses of the theorem with $\|\chi_A\|_{A(G)} \ll \log \epsilon^{-1}$; we conjecture that this construction represents the true state of affairs *viz.*

Conjecture 1.3. *Suppose that G is a compact vector space over \mathbb{F}_2 and $A \subset G$ has density α with $|\alpha - 1/3| \leq \epsilon$. Then*

$$\|\chi_A\|_{A(G)} \gg \log \epsilon^{-1}.$$

The paper splits into five further sections. Although the result of the introduction captures the spirit of this work, in §2 we explain our results in a more general and natural context. §3 then provides some examples which complement our results and are worth bearing in mind when following the proof. §4 is the central iterative argument; in this section we essentially prove a result with the conclusion of Theorem 1.2 but with a more cumbersome hypothesis on A . §5 then provides some physical space estimates to show that sets of density close to $1/3$ satisfy this hypothesis. The final section, §6, combines the work of the previous two sections to prove a result which implies the main result of the paper and discusses the limitations of our methods.

2. THE PROBLEM IN CONTEXT

Proposition 1.1 is, in fact, a special case of the following equally simple but more natural result essentially due to Cohen [Coh60]; a proof and more detailed discussion is contained in [San06].

Proposition 2.1. *Suppose that G is a compact Abelian group. Suppose that $A \subset G$ has density α and for all finite $V \leq \widehat{G}$ we have $\{\alpha|V|\}(1 - \{\alpha|V|\}) > 0$ where $\{\alpha|V|\}$ denotes the fractional part of $\alpha|V|$. Then $\chi_A \notin A(G)$.*

We are interested in quantitative versions of this proposition. Specifically we ask the following question.

Question 2.2. *Suppose that G is a compact Abelian group. Suppose that $A \subset G$ has density α and for all finite $V \leq \widehat{G}$ with $|V| \leq M$ we have $\{\alpha|V|\}(1 - \{\alpha|V|\}) \gg 1$. Then how small can $\|\chi_A\|_{A(G)}$ be in terms of M ?*

Green and Konyagin made progress on this question in [GK09] where they addressed the case when \widehat{G} has sparse small subgroup structure. They proved the following result.

Theorem 2.3. *Suppose that G is a compact Abelian group and the only subgroup $V \leq \widehat{G}$ with $|V| \leq M$ is the trivial group. Suppose that $A \subset G$ has density α and for all finite $V \leq \widehat{G}$ with $|V| \leq M$ we have $\{\alpha|V|\}(1 - \{\alpha|V|\}) \gg 1$. Then*

$$\|\chi_A\|_{A(G)} \gg \left(\frac{\log M}{\log \log M} \right)^{\frac{1}{3}}.$$

The density condition here collapses to $\alpha(1 - \alpha) \gg 1$ but we retain the more complex form for illustrative purposes. In this paper we address the complementary case when \widehat{G} has very rich small subgroup structure, specifically when G is a compact vector space over \mathbb{F}_2 . We prove the following result.

Theorem 2.4. *Suppose that G is a compact vector space over \mathbb{F}_2 . Suppose that $A \subset G$ has density α and for all finite $V \leq \widehat{G}$ with $|V| \leq M$ we have $\{\alpha|V|\}(1 - \{\alpha|V|\}) \gg 1$. Then*

$$\|\chi_A\|_{A(G)} \gg \log \log M.$$

The ideas of this paper and [GK09] can be combined to address the original question more fully. In [San06] we show the following result.

Theorem 2.5. *Suppose that G is a compact Abelian group. Suppose that $A \subset G$ has density α and for all finite $V \leq \widehat{G}$ with $|V| \leq M$ we have $\{\alpha|V|\}(1 - \{\alpha|V|\}) \gg 1$. Then*

$$\|\chi_A\|_{A(G)} \gg \log \log \log M.$$

The examples of §3 show that the bound in Theorem 2.4 cannot be better than a constant multiple of $\log M$, and there are easy examples mentioned by Green and Konyagin in [GK09] to show that the bound in Theorem 2.3 cannot be better than a constant multiple of $\log M$. In the absence of any better examples we make the following conjecture.

Conjecture 2.6. *Suppose that G is a compact Abelian group. Suppose that $A \subset G$ has density α and for all finite $V \leq \widehat{G}$ with $|V| \leq M$ we have $\{\alpha|V|\}(1 - \{\alpha|V|\}) \gg 1$. Then*

$$\|\chi_A\|_{A(G)} \gg \log M.$$

The various statements in the introduction follow easily from those of this section once one recalls that if G is a compact vector space over \mathbb{F}_2 and $V \leq \widehat{G}$ is finite then $|V|$ is a power of 2.

3. SETS WITH SMALL $A(G)$ -NORM

Throughout this section G is a compact vector space over \mathbb{F}_2 .

We address the question of how to construct subsets of G of a prescribed density whose characteristic function has small $A(G)$ -norm. This complements the main results of the paper.

Cosets are the simplest example of sets whose characteristic function has small $A(G)$ -norm: Recall that if $V \leq \widehat{G}$ then we write V^\perp for the *annihilator* of V i.e.

$$V^\perp := \{x \in G : \gamma(x) = 1 \text{ for all } \gamma \in V\}.$$

If $V \leq \widehat{G}$ is finite and $A = x + V^\perp$ then a simple calculation gives

$$\widehat{\chi_A}(\gamma) = \begin{cases} \gamma(x)|V|^{-1} & \text{if } \gamma \in V \\ 0 & \text{otherwise.} \end{cases}$$

It follows that $\|\chi_A\|_{A(G)} = 1$. A coset has density 2^{-d} for some integer d ; to produce a set with a density not of this form we take unions of cosets.

Suppose that we are given $\alpha \in [0, 1]$ a terminating binary number. Write

$$\alpha = \sum_{i=1}^k 2^{-d_i},$$

where the d_i are strictly increasing. If we can find a sequence of disjoint cosets A_1, \dots, A_k such that $\mu_G(A_i) = 2^{-d_i}$, then their union $A := \bigcup_{i=1}^k A_i$ has

$$(3.1) \quad \|\chi_A\|_{A(G)} = \left\| \sum_{i=1}^k \chi_{A_i} \right\|_{A(G)} \leq \sum_{i=1}^k \|\chi_{A_i}\|_{A(G)} = k$$

by the triangle inequality, and density

$$\mu_G(A) = \sum_{i=1}^k \mu_G(A_i) = \sum_{i=1}^k 2^{-d_i} = \alpha$$

since the elements of the union are disjoint. To produce such cosets we take $\{0_{\widehat{G}}\} = \Lambda_0 < \Lambda_1 < \dots < \Lambda_k \leq \widehat{G}$ a nested sequence of subspaces with $\dim \Lambda_i = d_i$. Choose a sequence of vectors $\{\gamma_i : 1 \leq i \leq k\}$ such that $\gamma_i \in \Lambda_i \setminus \Lambda_{i-1}$ for $1 \leq i \leq k$. It is easy to see that this sequence must be linearly independent so we may take a sequence $\{x_i : 1 \leq i \leq k-1\}$ such that

$$(3.2) \quad \gamma_j(x_i) = \begin{cases} 1 & \text{if } j \neq i \\ -1 & \text{if } j = i \end{cases}$$

for all $1 \leq i \leq k-1$. Put

$$A_i = x_1 + \dots + x_{i-1} + \Lambda_i^\perp.$$

First we note that $\mu_G(A_i) = 2^{-d_i}$ and second that the sets A_i are pairwise disjoint: Suppose $j > i$ and $x \in A_j$ then $x = x_1 + \dots + x_{j-1} + x'$ where $x' \in \Lambda_j^\perp$ so that

$$\gamma_i(x) = \gamma_i(x_1) \dots \gamma_i(x_{j-1}) \gamma_i(x').$$

$i < j$ so $\Lambda_i < \Lambda_j$ from which it follows that $\gamma_i(x') = 1$. Consequently $\gamma_i(x) = \gamma_i(x_i) = -1$ by (3.2). However if $x \in A_i$ then by a similar calculation $\gamma_i(x) = 1$.

It follows that A_1, \dots, A_k are disjoint cosets of the appropriate size and hence their union, $A := \bigcup_{i=1}^k A_i$, has density α and $\|\chi_A\|_{A(G)} \leq k$.

The first density we apply this general construction to is

$$\alpha = \frac{1}{4} + \frac{1}{16} + \dots + \frac{1}{4^k}.$$

The set A we produce has density α and the following two properties.

- (1) A satisfies the hypotheses of Theorem 2.4 with $M = 4^k - 1$: If $V \leq \widehat{G}$ and $|V| \leq M$ then $|V| = 2^d$ for some $d < k$ and

$$\frac{2}{3} \geq \sum_{i=\lfloor d/2 \rfloor + 1}^k 2^d \cdot 4^{-i} = \{\alpha 2^d\} \geq \frac{2^d}{4^{\lfloor d/2 \rfloor + 1}} \geq \frac{1}{4},$$

and hence $\{\alpha|V|\}(1 - \{\alpha|V|\}) \geq 1/12$.

- (2) $\|\chi_A\|_{A(G)} \asymp k$: $\|\chi_A\|_{A(G)} \leq k$ follows by construction; $\|\chi_A\|_{A(G)} \gg k$ is slightly more involved:

$$\widehat{\chi_{A_i}}(\gamma) = \begin{cases} 4^{-i} \gamma(x_1) \dots \gamma(x_{i-1}) & \text{if } \gamma \in \Lambda_i \\ 0 & \text{otherwise.} \end{cases}$$

Hence we can bound $|\widehat{\chi_A}|$ from below using the linearity of the Fourier transform.

$$|\widehat{\chi_A}(\gamma)| \geq 4^{-i} - \sum_{j=i+1}^k 4^{-j} \geq \frac{2}{3} \cdot 4^{-i} \text{ if } \gamma \in \Lambda_i \setminus \Lambda_{i-1},$$

and so

$$\|\chi_A\|_{A(G)} \geq \sum_{i=1}^k \frac{2}{3} 4^{-i} \cdot |\Lambda_i \setminus \Lambda_{i-1}| \geq \sum_{i=1}^k \frac{2}{3} 4^{-i} \cdot \frac{3}{4} 4^i = \frac{k}{2}.$$

The conclusion of Theorem 2.4 is that $\|\chi_A\|_{A(G)} \gg \log k$ which should be compared with the fact that actually $\|\chi_A\|_{A(G)} \asymp k$.

4. AN ITERATION ARGUMENT IN FOURIER SPACE

Throughout this section G is a compact vector space over \mathbb{F}_2 and $A \subset G$ has density α .

4.1. A trivial lower bound. Suppose that $\alpha > 0$. It is natural to try to bound $\|\chi_A\|_{A(G)}$ by a combination of Hölder's inequality and Plancherel's theorem:

$$(4.1) \quad \|\chi_A\|_{A(G)} \|\widehat{\chi_A}\|_\infty \geq \|\widehat{\chi_A}\|_2^2 = \|\chi_A\|_2^2;$$

non-negativity of χ_A means that $\widehat{\chi_A}(0_{\widehat{G}}) = \|\chi_A\|_1$ so

$$(4.2) \quad \|\chi_A\|_1 \geq \|\widehat{\chi_A}\|_\infty \geq \widehat{\chi_A}(0_{\widehat{G}}) = \|\chi_A\|_1 \Rightarrow \|\widehat{\chi_A}\|_\infty = \|\chi_A\|_1.$$

$\chi_A \equiv \chi_A^2$ so $\|\chi_A\|_2^2 = \|\chi_A\|_1 = \alpha$, which is positive, and hence (4.1) tells us that

$$(4.3) \quad \|\chi_A\|_{A(G)} \geq 1.$$

Taking $A = G$ shows that we can do no better than this, so for $\|\chi_A\|_{A(G)}$ to be large we require some sort of (further) restriction on what A may be. The restriction we take evolves out of modifying the idea above so that it becomes a step in an iteration.

4.2. A weak iteration lemma. A weakness in the above deduction is that we have no good upper bound for $\|\widehat{\chi_A}\|_\infty$. In fact, as we saw, $\|\widehat{\chi_A}\|_\infty$ is necessarily large because $\widehat{\chi_A}$ is large at the trivial character, however we know nothing about how large $\widehat{\chi_A}$ is at any other character, a fact which we shall now exploit.

Write f for the *balanced function* of χ_A i.e. $f = \chi_A - \alpha$. Then

$$\widehat{f}(\gamma) = \begin{cases} 0 & \text{if } \gamma = 0_{\widehat{G}} \\ \widehat{\chi_A}(\gamma) & \text{otherwise.} \end{cases}$$

Applying Hölder's inequality and Plancherel's theorem in the same way as before we have

$$(4.4) \quad \|\chi_A\|_{A(G)} \|\widehat{f}\|_\infty \geq \langle \widehat{\chi_A}, \widehat{f} \rangle = \langle \chi_A, f \rangle = \alpha - \alpha^2.$$

Now, fix $\epsilon > 0$ to be optimized later. If α is bounded away from 0 and 1 by an absolute constant then either $\|\chi_A\|_{A(G)} \gg \epsilon^{-1}$ or $\|\widehat{f}\|_\infty \gg \epsilon$. In the former case we are done (since $\|\chi_A\|_{A(G)}$ is large) and in the latter we have a non-trivial character at which $\widehat{\chi_A}$ is large; we should like to start building up a collection of such characters.

Suppose that $\Gamma \subset \widehat{G}$ is a collection of characters on which we know $\widehat{\chi_A}$ has large ℓ^1 -mass. We want to produce a superset Γ' of Γ by adding some more characters

which support a significant ℓ^1 -mass of $\widehat{\chi}_A$. To find characters outside Γ on which $\widehat{\chi}_A$ has large ℓ^1 -mass we might replace f with a function f_Γ (in analogy with the earlier replacement of χ_A by f) defined by inversion:

$$(4.5) \quad \widehat{f}_\Gamma(\gamma) = \begin{cases} 0 & \text{if } \gamma \in \Gamma \\ \widehat{\chi}_A(\gamma) & \text{otherwise.} \end{cases}$$

The problem with this is that for general Γ we can say very little about f_Γ . If $V \leq \widehat{G}$, however, then f_V has a particularly simple form:

$$f_V = \sum_{\gamma \in \widehat{G}} \widehat{\chi}_A(\gamma)(1 - \widehat{\mu}_{V^\perp}(\gamma))\gamma = \chi_A * (\delta - \mu_{V^\perp}) = \chi_A - \chi_A * \mu_{V^\perp} \text{ a.e.}$$

Now suppose that $\Gamma = V \leq \widehat{G}$. We want to try to add characters to V to get a superspace $V' \leq \widehat{G}$ with

$$\sum_{\gamma \in V'} |\widehat{\chi}_A(\gamma)| \text{ significantly larger than } \sum_{\gamma \in V} |\widehat{\chi}_A(\gamma)|.$$

We can use the idea in (4.4) to do this; replace f by f_V in that argument:

$$(4.6) \quad \|\chi_A\|_{A(G)} \|\widehat{f}_V\|_\infty \geq \langle \widehat{\chi}_A, \widehat{f}_V \rangle = \langle \chi_A, f_V \rangle.$$

Before, an easy calculation gave us $\langle \chi_A, f \rangle = \alpha(1 - \alpha)$. To compute $\langle \chi_A, f_V \rangle$ we have a slightly more involved calculation.

Lemma 4.3.

$$(4.7) \quad \|f_V\|_1 = 2\langle \chi_A, f_V \rangle.$$

Proof. μ_{V^\perp} is a probability measure so $0 \leq \chi_A * \mu_{V^\perp}(x) \leq 1$, hence $f_V(x) \leq 0$ for almost all $x \notin A$ and $f_V(x) \geq 0$ for almost all $x \in A$; consequently

$$\|f_V\|_1 = \int \chi_A f_V d\mu_G + \int (1 - \chi_A)(-f_V) d\mu_G = 2\langle \chi_A, f_V \rangle - \int f_V d\mu_G.$$

But $\int f_V d\mu_G = 0$ since $\int \chi_A d\mu_G = \int \chi_A * \mu_{V^\perp} d\mu_G$, so we are done. \square

It follows that

$$(4.8) \quad \|\chi_A\|_{A(G)} \|\widehat{f}_V\|_\infty \geq \frac{\|f_V\|_1}{2}.$$

So either $\|\chi_A\|_{A(G)} \geq \epsilon^{-1}$ or there is a character γ such that $|\widehat{f}_V(\gamma)| \geq \epsilon \|f_V\|_1/2$. By construction of f_V we have $\widehat{f}_V(\gamma') = 0$ if $\gamma' \in V$ so that $\gamma \notin V$ — γ is a genuinely new character. We let V' be the space generated by γ and V and have our first iteration lemma:

Lemma 4.4. (Weak iteration lemma) *Suppose that G is a compact vector space over \mathbb{F}_2 , that $V \leq \widehat{G}$ is finite and $A \subset G$. Suppose that $\epsilon \in (0, 1]$. Then either $\|\chi_A\|_{A(G)} \geq \epsilon^{-1}$ or there is a superspace V' of V with $\dim V' = \dim V + 1$ and for which*

$$\sum_{\gamma \in V'} |\widehat{\chi}_A(\gamma)| \geq \frac{\epsilon \|f_V\|_1}{2} + \sum_{\gamma \in V} |\widehat{\chi}_A(\gamma)|.$$

Iterating this lemma leads to the following proposition.

Proposition 4.5. *Suppose that G is a compact vector space over \mathbb{F}_2 and $A \subset G$ is such that for all $V \leq \widehat{G}$ with $|V| \leq M$ we have $\|f_V\|_1 \gg 1$. Then*

$$\|\chi_A\|_{A(G)} \gg \sqrt{\log M}.$$

We omit the proof (it is not difficult and all the ideas are contained in the proof of Proposition 4.9) since the hypotheses the proposition assumes on A are prohibitively strong; nevertheless we can make use of these ideas.

4.6. A stronger iteration lemma. The main weakness of the above approach is that each time we apply the weak iteration lemma to find characters supporting more ℓ^1 -mass of $\widehat{\chi_A}$ (assuming we are not in the case when $\|\chi_A\|_{A(G)}$ is automatically large) we do not find very much ℓ^1 -mass, in fact we find mass in proportion to $\|f_V\|_1$ which consequently has to be assumed large. We can improve this by adding to V not just one character at which $\widehat{f_V}$ is large but all such characters. This idea wouldn't work but for two essential facts.

- (1) There are a lot of characters at which $\widehat{f_V}$ is large, in that the characters at which $\widehat{f_V}$ is large actually support a large amount of the sum $\langle \widehat{\chi_A}, \widehat{f_V} \rangle$.
- (2) There is a result due to Chang which implies that the characters at which $\widehat{f_V}$ is large are contained in a space of relatively small dimension.

We record Chang's result, [Cha02], now. (In fact we record something slightly different from Chang's result. For more details see the appendix.)

Theorem 4.7. (Chang's theorem) *Suppose that G is a compact vector space over \mathbb{F}_2 , $f \in L^2(G)$ and $\epsilon \in (0, 1]$. Then there is a subspace W of \widehat{G} such that $\{\gamma : |\widehat{f}(\gamma)| \geq \epsilon \|f\|_1\} \subset W$ and*

$$\dim W \leq e\epsilon^{-2} \max\{\log(\|f\|_2^{-2} \|f\|_1^{-2}), 1\}.$$

We are in a position to show:

Lemma 4.8. (Iteration lemma) *Suppose that G is a compact vector space over \mathbb{F}_2 , that $V \leq \widehat{G}$ is finite, $A \subset G$ has $\|f_V\|_1 > 0$ and $\chi_A \in A(G)$. Then there is a non-negative integer s and a superspace V' of V such that*

$$\sum_{\gamma \in V'} |\widehat{\chi_A}(\gamma)| - \sum_{\gamma \in V} |\widehat{\chi_A}(\gamma)| \gg \left(\frac{4}{3}\right)^s$$

and

$$\dim V' - \dim V \ll 4^s \log \|f_V\|_1^{-1}.$$

Proof. By Plancherel's theorem we have

$$\sum_{\gamma \in \widehat{G}} \widehat{\chi_A}(\gamma) \overline{\widehat{f_V}(\gamma)} = \langle \chi_A, f_V \rangle = \frac{1}{2} \|f_V\|_1,$$

where the second equality is Lemma 4.3. To make use of this we apply the triangle inequality to the left hand side and get the driving inequality of the lemma

$$(4.9) \quad \frac{1}{2} \|f_V\|_1 \leq \sum_{\gamma \in \widehat{G}} |\widehat{\chi_A}(\gamma)| |\widehat{f_V}(\gamma)|.$$

Write \mathcal{L} for the set of characters at which $\widehat{f_V}$ is non-zero. Partition \mathcal{L} by a dyadic decomposition of the range of values of $|\widehat{f_V}|$. Specifically, for each non-negative integer s , let

$$\Gamma_s := \{\gamma \in \widehat{G} : 2^{-s}\|f_V\|_1 \geq |\widehat{f_V}(\gamma)| > 2^{-(s+1)}\|f_V\|_1\}.$$

For all characters γ we have $|\widehat{f_V}(\gamma)| \leq \|f_V\|_1$ and if $\gamma \in \mathcal{L}$ then $|\widehat{f_V}(\gamma)| > 0$ so certainly the Γ_s s cover \mathcal{L} ; they are clearly disjoint and hence form a partition of \mathcal{L} . Write L_s for the ℓ^1 -norm of $\widehat{\chi_A}$ supported on Γ_s :

$$L_s := \sum_{\gamma \in \Gamma_s} |\widehat{\chi_A}(\gamma)|.$$

The right hand side of (4.9) can now be rewritten using these definitions:

$$\begin{aligned} \sum_{\gamma \in \widehat{G}} |\widehat{\chi_A}(\gamma)| |\widehat{f_V}(\gamma)| &= \sum_{\gamma \in \mathcal{L}} |\widehat{\chi_A}(\gamma)| |\widehat{f_V}(\gamma)| \text{ by the definition of } \mathcal{L} \\ &= \sum_{s=0}^{\infty} \sum_{\gamma \in \Gamma_s} |\widehat{\chi_A}(\gamma)| |\widehat{f_V}(\gamma)| \text{ since } \{\Gamma_s\}_{s \geq 0} \text{ is a partition of } \mathcal{L}, \\ &\leq \sum_{s=0}^{\infty} \sum_{\gamma \in \Gamma_s} |\widehat{\chi_A}(\gamma)| \cdot 2^{-s} \|f_V\|_1 \text{ by the definition of } \Gamma_s, \\ &= \sum_{s=0}^{\infty} L_s 2^{-s} \|f_V\|_1 \text{ by the definition of } L_s. \end{aligned}$$

Combining this with (4.9) and dividing by $\|f_V\|_1$ (which is possible since $\|f_V\|_1 > 0$) we get

$$(4.10) \quad \frac{1}{2} \leq \sum_{s=0}^{\infty} 2^{-s} L_s.$$

Now, if for every non-negative integer s we have

$$L_s < \frac{1}{6} \left(\frac{4}{3}\right)^s,$$

then

$$\sum_{s=0}^{\infty} 2^{-s} L_s < \sum_{s=0}^{\infty} 2^{-s} \frac{1}{6} \left(\frac{4}{3}\right)^s = \frac{1}{6} \sum_{s=0}^{\infty} \left(\frac{2}{3}\right)^s = \frac{1}{2},$$

which contradicts (4.10). Hence there is a non-negative integer s such that

$$L_s \geq \frac{1}{6} \left(\frac{4}{3}\right)^s.$$

Chang's theorem gives a space W for which

$$\Gamma_s \subset \{\gamma \in \widehat{G} : |\widehat{f_V}(\gamma)| \geq 2^{-(s+1)}\|f_V\|_1\} \subset W$$

and

$$\dim W \leq 4e2^{2s} \max\{\log(\|f_V\|_2^2 \|f_V\|_1^{-2}), 1\}.$$

To tidy this up we note that $f_V = \chi_A - \chi_A * \mu_{V^\perp}$ a.e. and $\chi_A(x), \chi_A * \mu_{V^\perp}(x) \in [0, 1]$, so $f_V(x) \in [-1, 1]$ for a.e. $x \in G$ and hence $\|f_V\|_2 \leq 1$, from which it follows that

$$\dim W \ll 4^s \log \|f_V\|_1^{-1}.$$

Let V' be the space generated by V and W . Then

$$\dim V' - \dim V \ll 4^s \log \|f_V\|_1^{-1}.$$

Finally we note that $\Gamma_s \cap V = \emptyset$ since $\widehat{f_V}(\gamma) = 0$ if $\gamma \in V$ (recall $\widehat{f_V}$ from (4.5)) and $|\widehat{f_V}(\gamma)| > 2^{s+1} \|f_V\|_1 \geq 0$ if $\gamma \in \Gamma_s$. Hence

$$\sum_{\gamma \in V'} |\widehat{\chi_A}(\gamma)| \geq \sum_{\gamma \in \Gamma_s} |\widehat{\chi_A}(\gamma)| + \sum_{\gamma \in V} |\widehat{\chi_A}(\gamma)| \geq \frac{1}{6} \left(\frac{4}{3}\right)^s + \sum_{\gamma \in V} |\widehat{\chi_A}(\gamma)|.$$

This gives the result. \square

By iterating this lemma we prove the following result.

Proposition 4.9. *Suppose that G is a compact vector space over \mathbb{F}_2 and $A \subset G$ is such that for all $V \leq \widehat{G}$ with $|V| \leq M$ we have $\log \|f_V\|_1^{-1} \ll \log |V|$. Then*

$$\|\chi_A\|_{A(G)} \gg \log \log M.$$

Proof. Fix $\epsilon \in (0, 1]$ to be optimized later. We construct a sequence $V_0 \leq V_1 \leq \dots \leq \widehat{G}$ iteratively, writing $d_i := \dim V_i$ and

$$L_i = \sum_{\gamma \in V_i} |\widehat{\chi_A}(\gamma)|.$$

We start the construction by letting $V_0 := \{0_{\widehat{G}}\}$. Suppose that we are given V_k . If $|V_k| \leq M$ then apply the iteration lemma to V_k and A to get an integer s_{k+1} and vector space V_{k+1} with

$$(4.11) \quad d_{k+1} - d_k \ll 4^{s_{k+1}} \log \|f_{V_k}\|_1^{-1} \text{ and } L_{k+1} - L_k \gg \left(\frac{4}{3}\right)^{s_{k+1}}.$$

First we note that the iteration terminates since certainly $L_k \gg k$, but also $L_k \leq \|\chi_A\|_{A(G)} < \infty$.

Since $\log \|f_{V_k}\|_1^{-1} \ll \log |V_k| \ll d_k$ it follows from (4.11) that

$$(4.12) \quad d_{k+1} \ll 4^{s_{k+1}} d_k,$$

from which, in turn, we get

$$(4.13) \quad L_k \gg \sum_{l=0}^k \left(\frac{4}{3}\right)^{s_l} \gg \sum_{l=0}^k s_l \gg \log d_k.$$

Let K be the stage of the iteration at which it terminates i.e. $|V_K| > M$. We have two possibilities.

- (1) $d_{K-1} \equiv \log_2 |V_{K-1}| \leq \sqrt{\log M}$: in which case $d_K \geq \sqrt{\log M} \cdot d_{K-1}$. (4.12) then tells us that $4^{s_K} \gg \sqrt{\log M}$. However the first inequality in (4.13) tells us that $\|\chi_A\|_{A(G)} \geq L_K \gg (4/3)^{s_K}$ and so certainly $\|\chi_A\|_{A(G)} \gg \log \log M$.
- (2) Alternatively $d_{K-1} \equiv \log_2 |V_{K-1}| \geq \sqrt{\log M}$: in which case by (4.13) we have $L_{K-1} \gg \log d_{K-1} \gg \log \log M$ and so certainly $\|\chi_A\|_{A(G)} \gg \log \log M$.

In either case the proof is complete. \square

5. PHYSICAL SPACE ESTIMATES

To realize the hypothesis of Proposition 4.9 regarding f_V as a density condition we have the following lemma:

Lemma 5.1. *Suppose that G is a compact vector space over \mathbb{F}_2 , that $V \leq \widehat{G}$ is finite and $A \subset G$ has density α . Then*

$$\|f_V\|_1 \geq 2|V|^{-1}\{\alpha|V|\}(1 - \{\alpha|V|\}).$$

We need the following technical lemma:

Lemma 5.2. *Let $\delta_1, \dots, \delta_m \in [0, 1]$ and put $\gamma = \{\sum_{i=1}^m \delta_i\}$. Then*

$$(5.1) \quad \sum_{i=1}^m \delta_i - \delta_i^2 \geq \gamma(1 - \gamma).$$

Proof. We may assume that $0 < \gamma < 1$. Suppose that we have $i \neq j$ such that $0 < \delta_i, \delta_j < 1$. Put $\delta = \delta_i + \delta_j \leq 2$ and we have two cases:

- (1) $\delta \leq 1$: In this case we may replace δ_i and δ_j by δ and 0. This preserves γ and since

$$\delta_i - \delta_i^2 + \delta_j - \delta_j^2 \geq (\delta_i + \delta_j) - (\delta_i + \delta_j)^2 + 0 - 0^2,$$

it does not increase the sum in (5.1).

- (2) $2 \geq \delta > 1$: In this case we may replace δ_i and δ_j by 1 and $\delta - 1$. This preserves γ and since

$$\begin{aligned} (\delta_i - 1)(\delta_j - 1) &\geq 0 \\ \Rightarrow 0 &\geq -2\delta_i\delta_j + 2(\delta_i + \delta_j) - 2 \\ \Rightarrow \delta_i - \delta_i^2 + \delta_j - \delta_j^2 &\geq \delta_i - \delta_i^2 + \delta_j - \delta_j^2 - 2\delta_i\delta_j + 2(\delta_i + \delta_j) - 2 \\ \Rightarrow \delta_i - \delta_i^2 + \delta_j - \delta_j^2 &\geq (\delta_i + \delta_j - 1) - (\delta_i + \delta_j - 1)^2 + 1 - 1^2, \end{aligned}$$

it does not increase the sum in (5.1).

In both cases we can reduce the number of i s for which $0 < \delta_i < 1$ without increasing the sum in (5.1), so we may assume that there is only one j such that $0 < \delta_j < 1$. Then

$$\delta_j + \sum_{i \neq j} \delta_i = \gamma + \lfloor \sum_{i=1}^m \delta_i \rfloor \Rightarrow \delta_j - \gamma = \lfloor \sum_{i=1}^m \delta_i \rfloor - \sum_{i \neq j} \delta_i,$$

but the right hand side is an integer and $-1 < \delta_j - \gamma < 1$ so $\delta_j = \gamma$ and (5.1) follows. \square

of Lemma 5.1. Lemma 4.3 states that $\|f_V\|_1 = 2\langle \chi_A, f_V \rangle$ so

$$\begin{aligned} \|f_V\|_1 &= 2 \int \chi_A (\chi_A - \chi_A * \mu_{V^\perp}) d\mu_G \\ &= 2 \int_{x \in G} \int \chi_A (\chi_A - \chi_A * \mu_{V^\perp}) d\mu_{x+V^\perp} d\mu_G(x). \end{aligned}$$

(This is just conditional expectation.) $\chi_A * \mu_{V^\perp}$ is constant on cosets of V^\perp and $\chi_A^2 \equiv \chi_A$ so that

$$\begin{aligned} \|f_V\|_1 &= 2 \int_{x \in G} \int \chi_A d\mu_{x+V^\perp} (1 - \chi_A * \mu_{V^\perp}(x)) d\mu_G(x) \\ &= 2 \int_{x \in G} \chi_A * \mu_{V^\perp}(x) (1 - \chi_A * \mu_{V^\perp}(x)) d\mu_G(x). \end{aligned}$$

There are $|V|$ cosets of V^\perp in G , and $\chi_A * \mu_{V^\perp}$ is constant on cosets of V^\perp so this integral is really a finite sum with $|V|$ terms in it. Let \mathcal{C} be a set of coset representatives for V^\perp in G then $|\mathcal{C}| = |V|$ and

$$\|f_V\|_1 = \frac{2}{|\mathcal{C}|} \sum_{x' \in \mathcal{C}} \chi_A * \mu_{V^\perp}(x') (1 - \chi_A * \mu_{V^\perp}(x')).$$

We can now apply Lemma 5.2 to the $\chi_A * \mu_{V^\perp}(x')$ s with $m = |\mathcal{C}|$. This gives

$$\|f_V\|_1 \geq \frac{2}{|\mathcal{C}|} \beta(1 - \beta) = \frac{2}{|V|} \beta(1 - \beta)$$

where

$$\beta = \left\{ \sum_{x' \in \mathcal{C}} \chi_A * \mu_{V^\perp}(x') \right\} = \left\{ |\mathcal{C}| \int_{x \in G} \chi_A * \mu_{V^\perp}(x) d\mu_G(x) \right\} = \{|V|\alpha\}.$$

□

Nothing better than Lemma 5.1 can be true: Let A be the union of $\lfloor \alpha|V| \rfloor$ cosets of V^\perp and a subset of a coset of V^\perp of relative density $\{\alpha|V|\}$. Equality is attained in Lemma 5.1 for this set.

6. THE RESULT, REMARKS AND EXAMPLES

As an easy corollary of Proposition 4.9 and Lemma 5.1 we have:

Theorem 6.1. *Suppose that G is a compact vector space over \mathbb{F}_2 . Suppose that $A \subset G$ has density α and for all $V \leq \widehat{G}$ with $|V| \leq M$ we have $\{\alpha|V|\}(1 - \{\alpha|V|\}) \gg |V|^{-1}$. Then*

$$\|\chi_A\|_{A(G)} \gg \log \log M.$$

Theorem 2.4 is simply a weaker version of this result.

If we were interested we could read the explicit dependencies between the implied constants in Theorem 6.1 out of the proof. Since the only real importance of this result is in its corollary, Theorem 2.4, where even the M dependence is almost certainly not best possible, this is probably of little interest.

There are clear similarities between the methods of this paper and those employed by Green and Konyagin in [GK09], however the most striking ones seem to be between this work and the work of Bourgain in [Bou02]. In particular a slight variation on the calculation in Lemma 4.3 is in his work and he proves a result using Beckner's inequality (which is essentially equivalent to Chang's theorem) which shows that if $A \subset \mathbb{F}_2^n$ has density α with $\alpha(1 - \alpha) \gg 1$ then either $\widehat{\chi_A}$ is large at a non-trivial character or there is significant ℓ^2 -mass in the tail of the Fourier transform.

Our reason for stating Theorem 6.1 at all is that it is sharp up to the constant and hence demonstrates a limitation of our method. If we let

$$\alpha = \frac{1}{2^{2^0}} + \frac{1}{2^{2^1}} + \dots + \frac{1}{2^{2^{k-1}}},$$

then we showed in §3 that there is a set A of density α with $\|\chi_A\|_{A(G)} \leq k$. However A also satisfies the hypotheses of Theorem 6.1 with $M = 2^{2^{k-1}} - 1$: If $V \leq \widehat{G}$ has $|V| \leq M$ then $|V| = 2^d$ for some $d < 2^{k-1}$,

$$\begin{aligned} \{\alpha|V|\} &= \sum_{\min\{0, \log_2 d\} < m \leq k-1} 2^d \cdot 2^{-2^m} \leq \sum_{\min\{0, \log_2 d\} < m \leq k-1} 2^{-2^{m-1}} \\ &\leq \sum_{m=0}^{\infty} 2^{-2^m} \leq \frac{7}{8}, \end{aligned}$$

and

$$\{\alpha|V|\} = \sum_{\min\{0, \log_2 d\} < m \leq k-1} 2^d \cdot 2^{-2^m} \geq 2^d \cdot 2^{-2^{\lfloor \log_2 d \rfloor + 1}} \geq 2^{-d} = |V|^{-1}.$$

Hence

$$\{\alpha|V|\}(1 - \{\alpha|V|\}) \gg |V|^{-1}.$$

Theorem 6.1 then tells us that $\|\chi_A\|_{A(G)} \gg k$.

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APPENDIX A. CHANG'S THEOREM IN COMPACT VECTOR SPACES OVER \mathbb{F}_2

Green was the first to observe that the original proof of Chang's theorem in [Cha02] can easily be adapted to prove a result not just about characteristic functions of sets, but about arbitrary elements of $L^2(G)$. It is this result which we recorded as Chang's theorem (Theorem 4.7) above. We take the opportunity to provide a proof of this using Beckner's inequality which as Green has noted in [Gre02], can be used in place of Rudin's inequality when we are in the setting of compact vector spaces over \mathbb{F}_2 .

Suppose that G is a compact vector space over \mathbb{F}_2 . If Λ is a finite set of characters on G and $\eta \in [-1, 1]$ then we can define the Riesz product

$$p_\eta := \prod_{\lambda \in \Lambda} (1 + \eta\lambda).$$

Every term in this product is real and non-negative since $\eta \in [-1, 1]$, so p_η is real and non-negative.

Now suppose that Λ is linearly independent. Then $\widehat{p}_\eta(0_{\widehat{G}}) = 1$ and hence $\|p_\eta\|_1 = \widehat{p}_\eta(0_{\widehat{G}}) = 1$ by non-negativity of p_η . It follows by Young's inequality that we can define the operator

$$\begin{aligned} T_\eta : L^2(G) &\rightarrow L^2(G) \\ f &\mapsto f * p_\eta, \end{aligned}$$

and moreover $\|T_\eta f\|_2 \leq \|f\|_2 \|p_\eta\|_1 = \|f\|_2$. In fact there is a stronger inequality.

Theorem A.1. (Beckner's inequality) *Suppose that G is a compact vector space over \mathbb{F}_2 and Λ is a finite linearly independent set of characters on G . Suppose that $\eta \in [-1, 1]$. Then the operator T_η defined above has $\|T_\eta f\|_2 \leq \|f\|_{1+\eta^2}$.*

Having stated Beckner's inequality we are in a position to proceed with the proof of Chang's theorem. Chang noted the following simple fact regarding linearly independent sets.

Lemma A.2. *Suppose that V is a vector space, Γ is a subset of V and Λ is a maximal linearly independent subset of Γ . Then Γ is contained in the subspace generated by Λ .*

Theorem 4.7 now follows from this and the following result.

Proposition A.3. *Suppose that G is a compact vector space over \mathbb{F}_2 , $f \in L^2(G)$ and $\epsilon \in (0, 1]$. Suppose that Λ is a linearly independent subset of $\Gamma := \{\gamma : |\widehat{f}(\gamma)| \geq \epsilon \|f\|_1\}$. Then*

$$|\Lambda| \leq \epsilon \epsilon^{-2} \max\{\log(\|f\|_2^{-2} \|f\|_1^{-2}), 1\}$$

Proof. It certainly suffices to prove the result for Λ finite. In this case the operator T_η is defined and we can apply Beckner's inequality.

$$\sum_{\lambda \in \Lambda} |\eta \widehat{f}(\lambda)|^2 \leq \sum_{\gamma \in \widehat{G}} |\widehat{T_\eta f}(\gamma)|^2 = \|T_\eta f\|_2^2 \leq \|f\|_{1+\eta^2}^2.$$

The equality in the middle is by Parseval's theorem. Since $\Lambda \subset \Gamma$ it follows that

$$(\eta \epsilon \|f\|_1)^2 |\Lambda| \leq \sum_{\lambda \in \Lambda} |\eta \widehat{f}(\lambda)|^2 \leq \|f\|_{1+\eta^2}^2.$$

After some manipulation and using the log-convexity of $\|\cdot\|_p$ we get

$$|\Lambda| \leq \eta^{-2} \epsilon^{-2} \|f\|_1^{-2} \|f\|_{1+\eta^2}^2 \leq \epsilon^{-2} \eta^{-2} \left(\frac{\|f\|_2^2}{\|f\|_1^2} \right)^{\eta^2}.$$

Finally we optimize by putting $\eta^{-2} = \max\{\log(\|f\|_2^2 \|f\|_1^{-2}), 1\}$ to get the result. \square

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